

Weak group inverse

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Abstract

In this paper, we introduce a weak group inverse (called the WG inverse in the present paper) for square matrices of an arbitrary index, and give some of its characterizations and properties. Furthermore, we introduce two orders: one is a pre-order and the other is a partial order, and derive several characterizations of the two orders. At last, one characterization of the core-EP order is derived by using the WG inverses.

Keywords: group inverse; weak group inverse; WG order; core-EP order; C-E partial order; core-EP decomposition

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1. Introduction

In this paper, we use the following notations. The symbol $\mathbb{C}_{m,n}$ is the set of $m \times n$ matrices with complex entries; A^* , $\mathcal{R}(A)$ and $\text{rk}(A)$ represent the *conjugate transpose*, *range space* (or *column space*) and *rank* of $A \in \mathbb{C}_{m,n}$. Let $A \in \mathbb{C}_{n,n}$, the smallest positive integer k , which satisfies $\text{rk}(A^{k+1}) = \text{rk}(A^k)$, is called the *index* of A and is denoted as $\text{Ind}(A)$. The symbol \mathbb{C}_n^{CM} stands for the set of $n \times n$ matrices of index equal to one. The *Moore-Penrose inverse* of $A \in \mathbb{C}_{m,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying the equations:

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) (AX)^* = AX, \quad (4) (XA)^* = XA,$$

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and is denoted as $X = A^\dagger$; if X satisfies the equation $AXA = A$, then X is called a *g-inverse* of A , and is denoted as A^- ; E_A stands for the one orthogonal projection $E_A = I - AA^\dagger$. The *Drazin inverse* of $A \in \mathbb{C}_{n,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the equations

$$(1^k) \quad XA^{k+1} = A^k, \quad (2) \quad XAX = X, \quad (5) \quad AX = XA,$$

and is usually denoted as $X = A^D$. In particular, when $A \in \mathbb{C}_n^{\mathbf{CM}}$, the matrix X is called the *group inverse* of A , and is denoted as $X = A^\#$ (see [3]). The *core inverse* of $A \in \mathbb{C}_n^{\mathbf{CM}}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying

$$AX = AA^\dagger, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)$$

and is denoted as $X = A^\oplus$ [1]. When $A \in \mathbb{C}_n^{\mathbf{CM}}$, we call it a core invertible (or group invertible) matrix.

Recently, the research of the core inverse and related problems is drawing
5 ever-growing attention. Several generalized core inverses are introduced, which are the DMP inverse, the B-T inverse and the core-EP inverse [2, 9, 10]. Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$. The *DMP inverse* of A is $A^{d,\dagger} = A^D AA^\dagger$ [9]. The *B-T inverse* of A is $A^\diamond = (A^2 A^\dagger)^\dagger$ [2, Definition 1]. The *core-EP inverse* of A is $A^\oplus = A^k \left((A^*)^k A^{k+1} \right)^- A^k$ [10, Theorem 3.5 and Remark 2]. Especially, when
10 $A \in \mathbb{C}_n^{\mathbf{CM}}$, $A^\oplus = A^\diamond = A^{d,\dagger} = A^\oplus$ [2, 9, 10]. The relevant orders are presented, for example, the core-EP order, the DMP order and the B-T order [2, 6, 16]. The three orders are all pre-orders, although the core order is a partial order.

In [16], Wang introduced the core-EP decomposition. Applying the decomposition, Wang introduced the core-minus partial order, in a way similar to
15 applying the core-nilpotent decomposition to define the C-N partial order.

Furthermore, it is known that the index of group invertible matrix is also equal to one, that is, one matrix is core invertible if and only if it is group invertible. Although the generalizations of the core inverse attract much attention, the generalizations of the group inverse get little. Therefore, it is of interest to
20 inquire whether the group inverse can be generalized by some decompositions.

In this paper, our main tools are two decompositions: one is the core decomposition, the other is the core-EP decomposition. The aim of the paper is to introduce a generalized group inverse, consider its applications and derive some of its characterizations and properties.

25 2. Preliminaries

In this section, we present some preliminary results.

LEMMA 2.1. [3] *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$. Then*

$$A^D = A^k (A^{k+1})^\# . \quad (2.1)$$

LEMMA 2.2. [1, 7, 16] *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$. Then there exists a unitary matrix U such that*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \quad (2.2)$$

where $\Sigma = \text{diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \dots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A , $\sigma_1 > \sigma_2 > \dots > \sigma_t > 0$, $r_1 + r_2 + \dots + r_t = r$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy $KK^* + LL^* = I_r$.

Furthermore, A is core invertible if and only if ΣK is non-singular. When $A \in \mathbb{C}_n^{\text{CM}}$, (2.2) is called the core decomposition of A and

$$A^\oplus = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad (2.3)$$

$$A^\# = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \quad (2.4)$$

30 where $T = \Sigma K$ and $S = \Sigma L$.

It is well known that the core-nilpotent decomposition has been widely used in matrix theory [3, 8, 13]:

LEMMA 2.3. [13, Core-nilpotent decomposition] *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$, then A can be written as the sum of matrices \hat{A}_1 and \hat{A}_2 , i.e. $A = \hat{A}_1 + \hat{A}_2$, where*

$$\hat{A}_1 \in \mathbb{C}_n^{\text{CM}}, \quad \hat{A}_2^k = 0 \text{ and } \hat{A}_1 \hat{A}_2 = \hat{A}_2 \hat{A}_1 = 0.$$

Similarly, Wang introduced the notion of the core-EP decomposition in [16]:

LEMMA 2.4. [16, Core-EP Decomposition] *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$, then A can be written as the sum of matrices A_1 and A_2 , i.e. $A = A_1 + A_2$, where*

- (i) $A_1 \in \mathbb{C}_n^{\text{CM}}$;
- (ii) $A_2^k = 0$;
- (iii) $A_1^* A_2 = A_2 A_1 = 0$.

Here one or both of A_1 and A_2 can be null.

LEMMA 2.5. [16] *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in Lemma 2.4. Then there exists a unitary matrix U such that*

$$A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*, \quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \quad (2.5)$$

where T is non-singular, and N is nilpotent. Furthermore, the core-EP inverse of A is

$$A^\oplus = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (2.6)$$

3. WG inverse

In this section, we apply the core-EP decomposition to introduce a generalized group inverse (i.e. the WG inverse) and consider some characterizations of the generalized inverse.

3.1. Definition and properties of the WG inverse

Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$, and consider the system of equations

$$(2') \quad AX^2 = X, \quad (3^c) \quad AX = A^\oplus A. \quad (3.1)$$

Let the core-EP decomposition of A be as in (2.5). Then the core-EP inverse A^\oplus of A can be formed as:

$$A^\oplus = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (3.2)$$

Suppose that

$$X = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*. \quad (3.3)$$

Substituting (3.3) for X in (3.1) and applying (3.2), we derive

$$\begin{aligned} AX^2 - X &= U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-2} & T^{-3}S \\ 0 & 0 \end{bmatrix} U^* - U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = 0; \\ AX - A^\oplus A &= U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* - U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = 0. \end{aligned}$$

Therefore, (3.3) is the solution of the system to equations (3.1).

Furthermore, suppose that both X and \mathcal{X} satisfy (3.1), then

$$X = AX^2 = A^\oplus AX = A^\oplus A^\oplus A = A^\oplus A\mathcal{X} = A\mathcal{X}^2 = \mathcal{X},$$

that is, the solution to the system of equations (3.1) is unique. We have the following:

THEOREM 3.1. *The system of equations (3.1) is consistent and has a unique solution (3.3).*

DEFINITION 3.1. *Let $A \in \mathbb{C}_{n,n}$ be a matrix of index k . The WG inverse of A , denoted as $A^\mathfrak{W}$, is defined to be the solution to the system (3.1).*

REMARK 3.1. *When $A \in \mathbb{C}_n^{\mathbf{CM}}$, we have $A^\mathfrak{W} = A^\#$.*

REMARK 3.2. *In [4, Definition 1], the notion of weak Drazin inverse was given: let $A \in \mathbb{C}_{n,n}$ and $\text{Ind}(A) = k$, then X is a weak Drazin inverse of A if X satisfies (1^k) . Applying (3.3), it is easy to check that the WG inverse $A^\mathfrak{W}$ is a weak Drazin inverse of A .*

More details about the weak Drazin inverse can be seen in [4, 5, 15].

In the following example, we explain that the WG inverse is different from
 60 the Drazin, DMP, core-EP and B-T inverses.

EXAMPLE 3.1. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. It is easy to check that $\text{Ind}(A) = 2$,

the Moore-Penrose inverse A^\dagger and the Drazin inverse A^D are

$$A^\dagger = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the DMP inverse $A^{d,\dagger}$ and the B-T inverse A^\diamond are

$$A^{d,\dagger} = A^D A A^\dagger = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^\diamond = (A^2 A^\dagger)^\dagger = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the core-EP inverse A^\oplus and the WG inverse A^Ψ are

$$A^\oplus = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^\Psi = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2. Characterizations of the WG inverse

Let $A = \hat{A}_1 + \hat{A}_2$ be the core-nilpotent decomposition of $A \in \mathbb{C}_{n,n}$. Then $A^D = \hat{A}_1^\#$. Applying Lemma 2.4, (2.5) and (3.3), we have the following theorem.

THEOREM 3.2. *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in (2.5).*

Then

$$A^{\mathfrak{W}} = A_1^{\#} = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*. \quad (3.4)$$

Since

$$\begin{aligned} AA^{\oplus}A &= U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\ &= U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

and

$$(A^{\oplus})^2 = (A^2)^{\oplus} = U \begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

we have the following theorem:

THEOREM 3.3. *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$. Then*

$$A^{\mathfrak{W}} = (AA^{\oplus}A)^{\#} = (A^{\oplus})^2 A = (A^2)^{\oplus} A.$$

Let the core-EP decomposition of A be as in (2.5). Then

$$A^k = U \begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix} U^*, \quad A^{k+1} = U \begin{bmatrix} T^{k+1} & T\Phi \\ 0 & 0 \end{bmatrix} U^*, \quad (3.5)$$

where $\Phi = \sum_{i=1}^k T^{i-1}SN^{k-i}$. It follows that

$$\begin{aligned} A^k (A^{k+2})^{\oplus} A &= U \begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-(k+2)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = A^{\mathfrak{W}}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} (A^{k+2} (A^k)^{\dagger})^{\dagger} A &= (A^2 A^k (A^k)^{\dagger})^{\dagger} A \\ &= U \begin{bmatrix} T^2 & 0 \\ 0 & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = A^{\mathfrak{W}}. \end{aligned} \quad (3.7)$$

⁶⁵ Therefore, we have the following theorem.

THEOREM 3.4. *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$. Then*

$$A^{\mathfrak{W}} = A^k (A^{k+2})^{\oplus} A = (A^2 P_{A^k})^{\dagger} A.$$

It is known that the Drazin inverse is one generalization of the group inverse. We will see the similarities and differences between the Drazin inverse and the WG inverse from the following corollaries.

COROLLARY 3.5. *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$. Then*

$$\text{rk}(A^{\mathfrak{W}}) = \text{rk}(A^D) = \text{rk}(A^k).$$

It is well known that $(A^2)^D = (A^D)^2$, but the same is not true for the WG inverse. Applying the core-EP decomposition (2.5) of A , we have

$$A^2 = U \begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix} U^* \quad (3.8)$$

and

$$(A^2)^{\mathfrak{W}} = U \begin{bmatrix} T^{-2} & T^{-4}(TS + SN) \\ 0 & 0 \end{bmatrix} U^*, \quad (A^{\mathfrak{W}})^2 = U \begin{bmatrix} T^{-2} & T^{-3}S \\ 0 & 0 \end{bmatrix} U^*. \quad (3.9)$$

Therefore, $(A^2)^{\mathfrak{W}} = (A^{\mathfrak{W}})^2$ if and only if $T^{-4}(TS + SN) = T^{-3}S$. Since T is
70 invertible, we derive the following Corollary 3.6.

COROLLARY 3.6. *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in (2.5). Then $(A^2)^{\mathfrak{W}} = (A^{\mathfrak{W}})^2$ if and only if $SN = 0$.*

The commutativity is one of the main characteristics of the group inverse. The Drazin inverse has the characteristic, too. It is of interest to inquire whether the same is true or not for the WG inverse. Applying the core-EP decomposition

(2.5) of A , we have

$$AA^{\mathfrak{W}} = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*; \quad (3.10a)$$

$$A^{\mathfrak{W}}A = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{bmatrix} U^*. \quad (3.10b)$$

Therefore, we have the following Corollary 3.7.

COROLLARY 3.7. *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in (2.5).*

Then $AA^{\mathfrak{W}} = A^{\mathfrak{W}}A$ if and only if $SN = 0$.

Let $SN = 0$, then by applying Corollary 3.6 and Corollary 3.7, we derive

$$A^2 = U \begin{bmatrix} T^2 & TS \\ 0 & N^2 \end{bmatrix} U^*, \dots, A^k = U \begin{bmatrix} T^k & T^{k-1}S \\ 0 & 0 \end{bmatrix} U^*, A^{k+1} = U \begin{bmatrix} T^{k+1} & T^kS \\ 0 & 0 \end{bmatrix} U^*.$$

Let t be a positive integer. It follows from applying (2.1), (2.4) and (2.6) that

$$\begin{aligned} (A^{t+1})^{\oplus} &= U \begin{bmatrix} T^{-(t+1)} & 0 \\ 0 & 0 \end{bmatrix} U^*, \\ (A^{k+1})^{\#} &= (A^{k+1})^{\oplus} = U \begin{bmatrix} T^{-(k+1)} & T^{-(k+2)}S \\ 0 & 0 \end{bmatrix} U^*, \\ A^D &= (A^{k+1})^{\#} A^k = U \begin{bmatrix} T^{-(k+1)} & T^{-(k+2)}S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^k & T^{k-1}S \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = A^{\mathfrak{W}}. \end{aligned}$$

Therefore, we have the following Corollary 3.8..

COROLLARY 3.8. *Let $A \in \mathbb{C}_{n,n}$ be with $\text{Ind}(A) = k$, the core-EP decomposition of A be as in (2.5) and $SN = 0$. Then*

$$A^{\mathfrak{W}} = A^D = (A^{k+1})^{\oplus} A^k = (A^{t+1})^{\oplus} A^t,$$

where t is a positive integer.

4. Two Orders

A binary operation on a set S is said to be a *pre-order* on S if it is reflexive and transitive. If the pre-order is also anti-symmetric, we call it a *partial order* [13, Chap 1]. Let S_1 and S_2 be sets, and $S_2 \subseteq S_1$, then a partial order $\stackrel{1}{\leq}$ on S_1 is said to be *implied* by a partial order $\stackrel{2}{\leq}$ on S_2 if for $A, B \in S_2$,

$$A \stackrel{2}{\leq} B \Rightarrow A \stackrel{1}{\leq} B.$$

The expression $A \not\stackrel{2}{\leq} B$ means that A is not below B under the partial order $\stackrel{2}{\leq}$.

In [13, Definition 4.4.1 and Definition 4.4.17], the definitions of the *Drazin order* and the *C-N partial order* are given:

$$A \stackrel{D}{\leq} B : A, B \in \mathbb{C}_{n,n}, \hat{A}_1 \stackrel{\#}{\leq} \hat{B}_1, \quad (4.1)$$

$$A \stackrel{\#,-}{\leq} B : A, B \in \mathbb{C}_{n,n}, \hat{A}_1 \stackrel{\#}{\leq} \hat{B}_1 \text{ and } \hat{A}_2 \bar{\leq} \hat{B}_2, \quad (4.2)$$

80 in which $A = \hat{A}_1 + \hat{A}_2$ and $B = \hat{B}_1 + \hat{B}_2$ are the core-nilpotent decompositions of A and B , respectively. Similarly, in this section, we apply the core-EP decomposition to introduce two orders: one is the WG order and the other is the C-E order.

4.1. WG order

Consider the binary operation:

$$A \stackrel{\text{WG}}{\leq} B : A, B \in \mathbb{C}_{n,n}, A_1 \stackrel{\#}{\leq} B_1, \quad (4.3)$$

85 in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , respectively.

Reflexivity of the relation is obvious. Suppose $A \stackrel{\text{WG}}{\leq} B$ and $B \stackrel{\text{WG}}{\leq} C$, in which $A = A_1 + A_2$, $B = B_1 + B_2$ and $C = C_1 + C_2$ are the core-EP decompositions of A , B and C , respectively. Then $A_1 \stackrel{\#}{\leq} B_1$ and $B_1 \stackrel{\#}{\leq} C_1$. Therefore $A_1 \stackrel{\#}{\leq} C_1$. It
90 follows from (4.3) that $A \stackrel{\text{WG}}{\leq} C$.

EXAMPLE 4.1. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Although $A \stackrel{\text{WG}}{\leq} B$ and $B \stackrel{\text{WG}}{\leq} A$, $A \neq B$. Therefore, the anti-symmetry of the binary operation (4.3) cannot be tenable.

Therefore, we have the following Theorem 4.1.

THEOREM 4.1. *The binary operation (4.3) is a pre-order. We call this pre-*
95 *order the WG order.*

REMARK 4.1. *The WG order coincides with the sharp partial order on \mathbb{C}_n^{CM} .*

In the following two examples, we see some differences between the WG order and the Drazin order.

EXAMPLE 4.2. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^D = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $A \stackrel{\text{WG}}{\leq} B$.

100 Since $A^D A \neq A^D B$, we derive $A \not\stackrel{D}{\leq} B$. Therefore, the WG order does not imply the Drazin order.

EXAMPLE 4.3. *Let*

$$\begin{aligned}\widehat{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \widehat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ A &= P\widehat{A}P^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = P\widehat{B}P^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = 0, B_1 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\end{aligned}$$

in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , respectively. Then $A \stackrel{D}{\leq} B$ and $A_1 \not\stackrel{\#}{\leq} B_1$. Therefore, the Drazin order does not imply the WG order.

105 It is well known that $A \stackrel{D}{\leq} B \Rightarrow A^2 \stackrel{D}{\leq} B^2$, but the same is not true for the WG order as the following example shows:

EXAMPLE 4.4. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We derive $A^2 \not\stackrel{\text{WG}}{\leq} B^2$. Therefore, $A \stackrel{\text{WG}}{\leq} B \not\Rightarrow A^2 \stackrel{\text{WG}}{\leq} B^2$.

Let $A \stackrel{\text{WG}}{\leq} B$, $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , A_1 and A_2 be as given in (2.5), and partition

$$U^* B_1 U = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \quad (4.4)$$

Applying (3.10a) and (3.10b) gives

$$\begin{aligned} A_1 A_1^\# &= U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*; \\ B_1 A_1^\# &= U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} B_{11}T^{-1} & B_{11}T^{-2}S \\ B_{21}T^{-1} & B_{21}T^{-2}S \end{bmatrix} U^*. \end{aligned}$$

Since $A \stackrel{\text{WG}}{\leq} B$, $A_1 \stackrel{\#}{\leq} B_1$. It follows from $A_1 A_1^\# = B_1 A_1^\#$ that

$$B_{11} = T \text{ and } B_{21} = 0. \quad (4.5)$$

By applying (4.4) and (4.5), we have

$$\begin{aligned} A_1^\# A_1 &= U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*, \\ A_1^\# B_1 &= U \begin{bmatrix} I & T^{-1}B_{12} + T^{-2}SB_{22} \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

It follows from $A_1^\# A_1 = A_1^\# B_1$ that

$$\begin{aligned} 0 &= T^{-1}S - (T^{-1}B_{12} + T^{-2}SB_{22}) \\ &= T^{-1}(S - T^{-1}SB_{22} - B_{12}). \end{aligned}$$

Therefore,

$$B_{12} = S - T^{-1}SB_{22}, \quad (4.6)$$

in which B_{22} is an arbitrary matrix of an appropriate size. From (4.5) and (4.6), we obtain

$$B_1 = U \begin{bmatrix} T & S - T^{-1}SB_{22} \\ 0 & B_{22} \end{bmatrix} U^*. \quad (4.7)$$

Since B_1 is core invertible, B_{22} is core invertible. Let the core decomposition of B_{22} be as

$$B_{22} = U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^*, \quad (4.8)$$

where T_1 is invertible. Denote

$$\widehat{U} = U \begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix}.$$

It is easy to see that \widehat{U} is a unitary matrix. Let SU_1 be partitioned as follows:

$$SU_1 = \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix}.$$

Then

$$A_1 = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^* \quad (4.9)$$

and

$$\begin{aligned} B_1 &= U \begin{bmatrix} T & S - T^{-1}SB_{22} \\ 0 & U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^* \end{bmatrix} U^* \\ &= U \begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} T & SU_1 - T^{-1}SU_1U_1^*B_{22}U_1 \\ 0 & \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U_1^* \end{bmatrix} U^* \\ &= \widehat{U} \begin{bmatrix} T & \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix} - T^{-1} \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix} \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \widehat{U}^* \\ &= \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*. \end{aligned} \quad (4.10)$$

From (4.3), (4.9) and (4.10), we derive the following Theorem 4.2.

THEOREM 4.2. *Let $A, B \in \mathbb{C}_{n,n}$. Then $A \stackrel{\text{wg}}{\leq} B$ if and only if there exists a*

unitary matrix \widehat{U} such that

$$A = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix} \widehat{U}^*, \quad (4.11a)$$

$$B = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 \end{bmatrix} \widehat{U}^*, \quad (4.11b)$$

where T and T_1 are invertible, $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ and N_2 are nilpotent.

4.2. C-E partial order

Consider the binary operation:

$$A \stackrel{\text{CE}}{\leq} B : A, B \in \mathbb{C}_{n,n}, A_1 \stackrel{\#}{\leq} B_1 \text{ and } A_2 \stackrel{-}{\leq} B_2, \quad (4.12)$$

in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , respectively.

DEFINITION 4.1. Let $A, B \in \mathbb{C}_{n,n}$. We say that A is below B under the C-E order if A and B satisfy the binary operation (4.12).

When A is below B under the C-E order, we write $A \stackrel{\text{CE}}{\leq} B$.

THEOREM 4.3. The C-E order is a partial order.

Proof. Reflexivity is trivial.

Let $A \stackrel{\text{CE}}{\leq} B \stackrel{\text{CE}}{\leq} C$ and $A = A_1 + A_2$, $B = B_1 + B_2$ and $C = C_1 + C_2$ are the core-EP decompositions of A , B and C , respectively. Then $A_1 \stackrel{\#}{\leq} B_1$, $B_1 \stackrel{\#}{\leq} C_1$ and $A_2 \stackrel{-}{\leq} B_2$, $B_2 \stackrel{-}{\leq} C_2$. Therefore $A_1 \stackrel{\#}{\leq} C_1$ and $A_2 \stackrel{-}{\leq} C_2$. It follows from Definition 4.1 that $A \stackrel{\text{CE}}{\leq} C$.

If $A \stackrel{\text{CE}}{\leq} B$ and $B \stackrel{\text{CE}}{\leq} A$, Then $A_1 = B_1$ and $A_2 = B_2$, that is, $A = B$. \square

THEOREM 4.4. *Let $A, B \in \mathbb{C}_{n,n}$. Then $A \stackrel{\text{CE}}{\leq} B$ if and only if there exists a unitary matrix U satisfying*

$$A = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & N_{22} \end{bmatrix} \widehat{U}^*, \quad (4.13a)$$

$$B = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 \end{bmatrix} \widehat{U}^*, \quad (4.13b)$$

where T and T_1 are invertible, N_{22} and N_2 are nilpotent, and $N_{22} \bar{\leq} N_2$.

Proof. Let $A \stackrel{\text{CE}}{\leq} B \stackrel{\text{CE}}{\leq} C$ and $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , respectively. Then $A_1 \stackrel{\#}{\leq} B_1$ and $A_2 \bar{\leq} B_2$. It follows from Theorem 4.2 and $A_2 \bar{\leq} B_2$ that $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \bar{\leq} \begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}$. Since

$$\text{rk}(N_{22}) \leq \text{rk} \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right)$$

and

$$\begin{aligned} \text{rk}(N_2) - \text{rk}(N_{22}) &\leq \text{rk}(N_2 - N_{22}) \leq \text{rk} \left(\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right) \\ &= \text{rk}(N_2) - \text{rk} \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right), \end{aligned}$$

we obtain

$$\text{rk}(N_{22}) = \text{rk} \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right), \quad N_{22} \bar{\leq} N_2, \quad (4.14)$$

and

$$\text{rk}(N_2) - \text{rk}(N_{22}) = \text{rk}(N_2) - \text{rk} \left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right). \quad (4.15)$$

Since $N_{22} \leq N_2$, there exist nonsingular matrices P and Q such that

$$N_{22} = P \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q, \quad N_2 = P \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q,$$

where D_1 and D_2 are nonsingular, (see [13, Theorem 3.7.3]). It follows that

$$\text{rk}(N_2) - \text{rk}(N_{22}) = \text{rk}(D_2). \quad (4.16)$$

Denote

$$N_{12} = \begin{bmatrix} M_{12} & M_{13} & M_{14} \end{bmatrix} Q, \text{ and } N_{21} = P \begin{bmatrix} M_{21} \\ M_{31} \\ M_{41} \end{bmatrix}. \quad (4.17)$$

Then

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I_{\text{rk}(N_{11})} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} N_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & D_1 & 0 & 0 \\ M_{31} & 0 & 0 & 0 \\ M_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\text{rk}(N_{11})} & 0 \\ 0 & Q \end{bmatrix}.$$

It follows from (4.14) that

$$M_{13} = 0, M_{14} = 0, M_{31} = 0 \text{ and } M_{41} = 0. \quad (4.18)$$

Therefore,

$$\begin{bmatrix} -N_{11} & -N_{12} \\ -N_{21} & N_2 - N_{22} \end{bmatrix} = \begin{bmatrix} I_{\text{rk}(N_{11})} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} -N_{11} & -M_{12} & 0 & 0 \\ -M_{21} & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\text{rk}(N_{11})} & 0 \\ 0 & Q \end{bmatrix}.$$

By applying (4.15), (4.16) and $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \overset{-}{\leq} \begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}$, we derive that

$$\begin{aligned} \text{rk} \left(\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \right) &= \text{rk} \left(\begin{bmatrix} N_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix} \right) + \text{rk}(D_2) \\ &= \text{rk}(N_2) - \text{rk}(N_{22}) \\ &= \text{rk}(D_2). \end{aligned}$$

¹²⁵ Therefore, $N_{11} = 0$, $M_{12} = 0$ and $M_{21} = 0$. By applying (4.17) and (4.18), we obtain $N_{11} = 0$, $N_{12} = 0$ and $N_{21} = 0$.

Let A and B be of the forms as given in (4.13a) and (4.13b), then $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B , respectively, and

$$\begin{aligned} A_1 &= \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*, \quad A_2 = \widehat{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{22} \end{bmatrix} \widehat{U}^*; \\ B_1 &= \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*, \quad B_2 = \widehat{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} \widehat{U}^*. \end{aligned}$$

It is easy to check that $A_1 \overset{\#}{\leq} B_1$ and $A_2 \overset{-}{\leq} B_2$. Therefore, $A \overset{\text{CE}}{\leq} B$. \square

REMARK 4.2. In Ex. 4.3, it is easy to check that $A \overset{\#,-}{\leq} B$. Since $A_1 \not\overset{\#}{\leq} B_1$, we have $A \not\overset{\text{CE}}{\leq} B$. Therefore, the C - N partial order does not imply the C - E partial order.

¹³⁰

COROLLARY 4.5. Let $A, B \in \mathbb{C}_{n,n}$. If $A \overset{\text{CE}}{\leq} B$, then $A \overset{-}{\leq} B$.

Proof. Let $A, B \in \mathbb{C}_{n,n}$. Then A and B have the forms as given in Theorem

4.4. Since T and T_1 are invertible, it follows that

$$\begin{aligned}
\text{rk}(B) &= \text{rk}(T) + \text{rk}(T_1) + \text{rk}(N_2); \\
\text{rk}(A) &= \text{rk}(T_1) + \text{rk}(N_{22}); \\
\text{rk}(B - A) &= \text{rk} \left(\begin{bmatrix} 0 & -T^{-1}\widehat{S}_1T_1 & -T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 - N_{22} \end{bmatrix} \right) \\
&= \text{rk} \left(\begin{bmatrix} I_{\text{rk}(T)} & T^{-1}\widehat{S}_1 & 0 \\ 0 & I_{\text{rk}(T_1)} & 0 \\ 0 & 0 & I_{n-\text{rk}(T)-\text{rk}(T_1)} \end{bmatrix} \begin{bmatrix} 0 & -T^{-1}\widehat{S}_1T_1 & -T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 - N_{22} \end{bmatrix} \right) \\
&= \text{rk} \left(\begin{bmatrix} T_1 & S_1 \\ 0 & N_2 - N_{22} \end{bmatrix} \right) = \text{rk} \left(\begin{bmatrix} T_1 & 0 \\ 0 & N_2 - N_{22} \end{bmatrix} \right) \\
&= \text{rk}(T_1) + \text{rk}(N_2 - N_{22}) \\
&= \text{rk}(T_1) + \text{rk}(N_2) - \text{rk}(N_{22}).
\end{aligned}$$

Therefore, $\text{rk}(B - A) = \text{rk}(B) - \text{rk}(A)$, that is, $A \stackrel{\bar{}}{\leq} B$. \square

5. Characterizations of the core-EP order

As is noted in [16], the core-EP order is given:

$$A \stackrel{\oplus}{\leq} B : A, B \in \mathbb{C}_{n,n}, A^{\oplus}A = A^{\oplus}B \text{ and } AA^{\oplus} = BA^{\oplus}. \quad (5.1)$$

Some characterizations of the core-EP order are given in [16].

LEMMA 5.1. [16] *Let $A, B \in \mathbb{C}_{n,n}$ and $A \stackrel{\oplus}{\leq} B$. Then there exists a unitary matrix U such that*

$$A = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2 \end{bmatrix} U^*, \quad (5.2)$$

135 where $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ and N_2 are nilpotent, T_1 and T_3 are non-singular.

Let the core-EP decomposition of A be as given in (2.5), and denote

$$U^*BU = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}. \quad (5.3)$$

By applying (3.10a) and

$$BA^\mathbb{W} = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} B_1T^{-1} & B_1T^{-2}S \\ B_3T^{-1} & B_3T^{-2}S \end{bmatrix} U^*,$$

we have $AA^\mathbb{W} = BA^\mathbb{W}$ if and only if

$$B_1 = T \text{ and } B_3 = 0.$$

It follows that

$$\begin{aligned} A^*A^\mathbb{W} &= U \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^*T^{-1} & T^*T^{-2}S \\ S^*T^{-1} & S^*T^{-2}S \end{bmatrix} U^*, \\ B^*A^\mathbb{W} &= U \begin{bmatrix} T^* & 0 \\ B_2^* & B_4^* \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^*T^{-1} & T^*T^{-2}S \\ B_2^*T^{-1} & B_2^*T^{-2}S \end{bmatrix} U^*. \end{aligned}$$

Therefore, $AA^\mathbb{W} = BA^\mathbb{W}$ and $A^*A^\mathbb{W} = B^*A^\mathbb{W}$ if and only if

$$B_1 = T, B_3 = 0, B_2 = S, \text{ and } B_4 \text{ is arbitrary}, \quad (5.4)$$

that is, A and B satisfy $AA^\mathbb{W} = BA^\mathbb{W}$ and $A^*A^\mathbb{W} = B^*A^\mathbb{W}$ if and only if there exists a unitary matrix U such that

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T & S \\ 0 & B_4 \end{bmatrix} U^*, \quad (5.5)$$

where N is nilpotent, T is non-singular and B_4 is arbitrary. Therefore, by applying Lemma 5.1, we derive one characterization of the core-EP order.

THEOREM 5.2. *Let $A, B \in \mathbb{C}_{n,n}$. Then $A \stackrel{\oplus}{\leq} B$ if and only if*

$$AA^\mathbb{W} = BA^\mathbb{W} \text{ and } A^*A^\mathbb{W} = B^*A^\mathbb{W}.$$

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